This document demonstrates some of the features of FIGPUT. It consists of a series of snippets of exposition. There are a few simple examples – Bézier curves, the ellipse and diffiusion – followed by a more extended example about gears.

1 Example: Bézier Curves

Bézier curves are a convenient way of defining curves in the plane. They can be defined using Bernstein polynomials. The Bernstein polynomials of degree n are defined by

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \qquad i = 0, \ldots, n.$$

A cubic Bézier curve is a linear combination of the Bernstein polynomials of degree 3:

$$C(t) = \sum_{i=0}^{3} p_i B_i^3(t),$$

where the four p_i are points in the plane and t is limited to $[0,1]$. This can be written as the pair of equations $(x(t), y(t)) = C(t)$, where

$$\begin{aligned} x(t) &= x_1 \cdot (1-t)^3 + x_2 \cdot 3(1-t)^2 t + x_3 \cdot 3(1-t)t^2 + x_4 \cdot t^3 \\ y(t) &= y_1 \cdot (1-t)^3 + y_2 \cdot 3(1-t)^2 t + y_3 \cdot 3(1-t)t^2 + y_4 \cdot t^3, \end{aligned}$$

$(x_i, y_i) = p_i$ and the indices on the p_i have been shifted. We have $C(0) = p_1$ and $C(1) = p_4$ and the other two points act as "controls." See Figure (1), noting that the line determined by the pair of points controlling each end of the curve is tangent to the corresponding end-point of the Bézier curve.

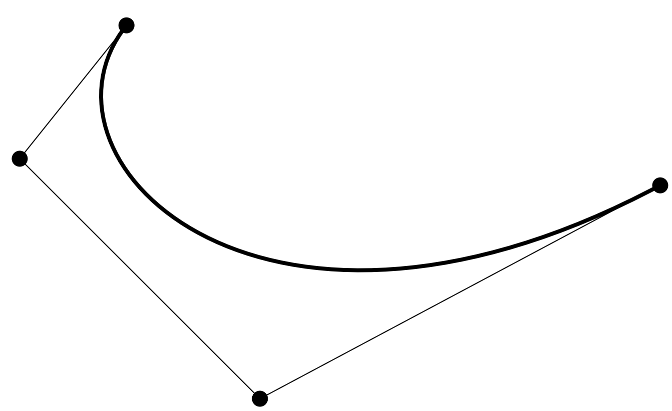

Figure 1: A Cubic Bézier Curve

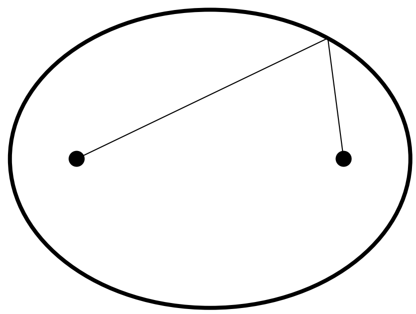

Figure 2: Tacks-and-string Ellipse

2 Example: Drawing an Ellipse

One way to define an ellipse is illustrated by Figure (2). Choose two points, F_1 and F_2 (the **foci**), and fix some $k > 0$. The ellipse is then the set of points, P, satisfying

$$d(P, F_1) + d(P, F_2) = k,$$

where $d(A, B)$ is the distance from A to B.

The figure makes it clear why this is sometimes refered to as the "tacks and string" definition. Imagine tacking each end of a bit of string, of length k, to the two foci; then tracing out the ellipse by holding a pencil at the limit of what the string will allow as the pencil travels about the foci.

3 Example: Diffusion

The concept of diffusion is illustrated by Figure (3). Statistical mechanics and Boltzmann's equation explain concepts like heat transfer and the gas laws by modeling the random motion of many particles. These equations may be difficult to grasp, but an intuitive understanding is not difficult. At time zero, there is some set of particles on the left half of the box, each of which is moving with some randomly distributed momentum. The divider is removed and, over time, the particles distribute themselves more evenly throughout the box.

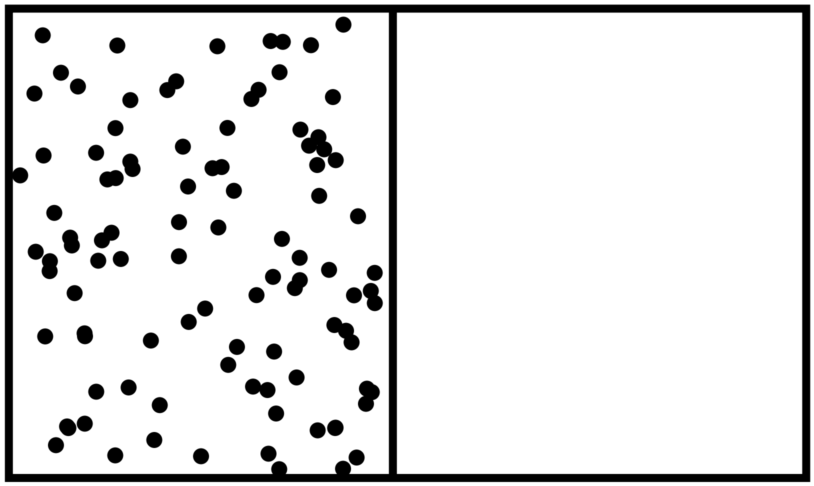

Figure 3: Diffusion and Boltzmann's Equation

4 Example: Gears

Gears like those in Figure (4) might work after a fashion, but it's rare to see such gears in anything other than a child's toy. As a rule, gears take the form shown in Figure (15). Why?

Figure (5) illustrates the fundamental problem of gear design, using a particularly bad design. The "gears" here have been reduced to something more like spokes. The gear on the right rotates counter-clockwise at a constant rate and drives the gear on the left. As the gears rotate, the rate of rotation of the gear on the left will not be constant; at some positions, the left gear is nearly stationary and at other positions, it rotates much faster than the gear on the right. In addition, the point of contact slides, generating friction and wear.

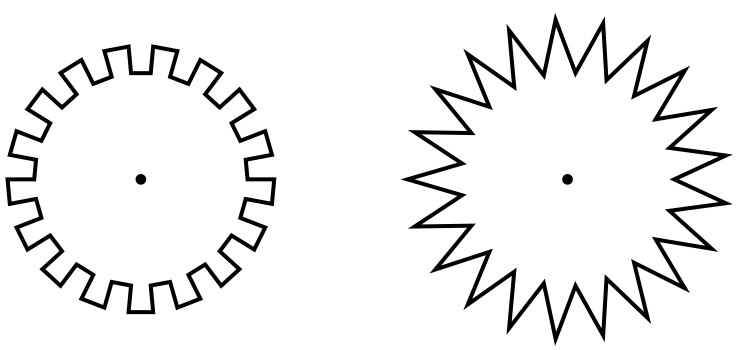

Figure 4: Crude Gear Designs

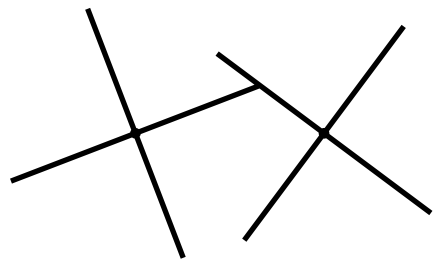

Figure 5: Maybe the Worst Possible Gears?

Understanding how to design gears without the kind of surging motion and wear inherent in the gears of Figure (5) explains why nearly all modern gears take the form of Figure (15). The most glaring problem with the spoke-like gears is the way their motion varies – imagine riding in a car with a gear-train based on such gears!

Gears typically have what's called *conjugate action*, meaning that the ratio of their rates of rotation is constant, with no surging or lagging. Sometimes this is also called the *fundamental law of gearing*, although it would be more accurate to call it a "commonly desired feature," rather than a "law." Arranging the teeth of gears so that they have conjugate action is surprisingly tricky.

4.1 The Involute

Figure (6) shows a cam and a lever-arm pushing against each other, causing them to rotate about their respective axes. Imagine that the cam rotates counter-clockwise, pushing the arm downwards. There are several crucial observations:

1. The two curves must be tangent at the point of contact.
2. The force from one part to the other must be directed along a line perpendicular to the two curves at the point of contact. Call this the *line of action*.
3. Let P be point where the line of action intersects the line connecting the two centers of rotation. This is called the *pitch point*. The instantaneous ratio of the two rates of rotation is equal to the ratio of the distances from P to each of the centers of rotation.

In conclusion, if two gears are to have conjugate action, then the pitch point must be fixed.

See Figure (7). Imagine a string wrapped around the circle, with one end fixed to the circle and a pencil at the other end. As the string unwraps from the circle, the pencil traces out a curve called the involute. The line determined by the string is obviously tangent to the circle at the point at which it meets the

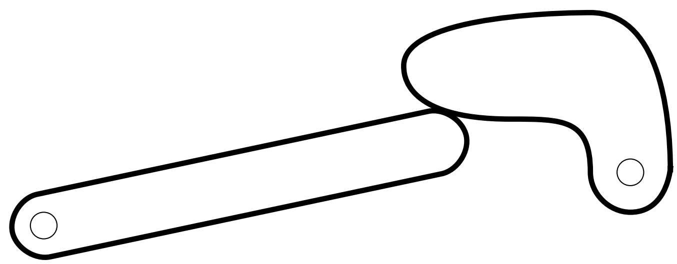

Figure 6: Basic Constraints on Gears

circle. The line is also perpendicular to the involute because, at each instantaneous point of rotation, the involute is locally an arc of the circle formed by the string. For a circle of radius r centered at the origin, the involute can be parameterized by $i(t) = (i_x(t), i_y(t))$, where

$$\begin{aligned} i_x(t) &= r(\cos t + t \sin t) \\ i_y(t) &= r(\sin t - t \cos t) \end{aligned}$$

As we will see, the notable thing about the involute is that when gear teeth take the form of an involute, the pitch point is constant throughout the gears' motion.

Figure (8) shows two disks acting as gears by simple friction. The circle of each such disk is called the *pitch circle* (where they come in contact is the pitch point). The centers of these disks are joined by the *line of centers*, and the distance between these centers is the *center distance*. Now imagine two slightly smaller and concentric circles, called the *base circles*. These base circles will be used to form involutes, and these involutes will be the profiles of the gear teeth.

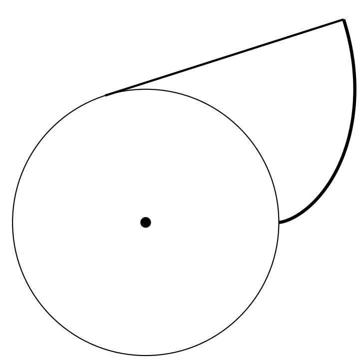

Figure 7: The Involute of a Circle

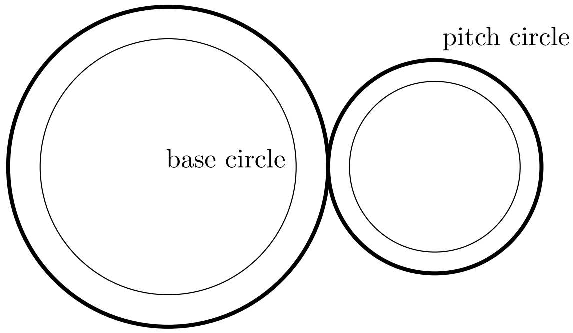

Figure 8: Base Circle and Pitch Circle

Figure (9) shows an enlarged view of the two base circles of Figure (8), without the pitch circles. Imagine that a piece of string is tightly wrapped around one base circle, extends over to the other base circle, and is wrapped around it too. As the disks rotate, the string unwinds from one base circle and is taken up by the other base circle. There is a fixed point on the string that represents the point of contact between two teeth. This point traces a path that is an involute relative to either circle. These involutes define the shape of the mating tooth profiles. The line of action is coincident with the string, and the two gears have conjugate action.

A fortunate feature of the involute is that the teeth can be truncated at

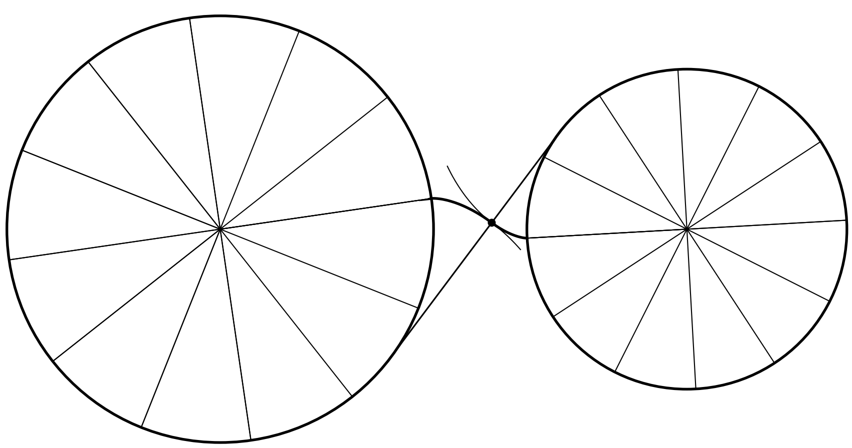

Figure 9: Rotation Traces Involutes

their perimeter, or their widths may be varied, yet the two gears still have conjugate action; the teeth come into contact sooner or later as they rotate, but the point of contact follows the same line of action. If the center distance changes, then the line of action also changes, but the tooth form (the involute) remains the same, and the gears still have conjugate action, though there will be some backlash and additional friction between the teeth.

What remains is the resolution of many practical issues: the interplay of gear radius, tooth size, number of teeth and the like.

The two main methods of gear specification are metric (ISO) and inch (AGMA), and the two systems use slightly different fundamental quantities to specify a given gear. These are the basic parameters used to specify off-the-shelf gear profiles.

- ϕ, pressure angle
- N, number of teeth
- m, module (for metric gears)
- p_d, diametral pitch (for inch gears)

There are many additional terms and measurements. In fact, off-the-shelf gears are typically specified in a way that makes various assumptions. Additional parameters that influence gear design are

- p_c, circular pitch
- d, pitch diameter
- r_p, pitch radius
- r_b, base radius
- a, addendum
- b, dedendum

The *pressure angle*, ϕ, is the angle between the line of action and the perpendicular to the line of centers. In Figure (9), the pressure angle is roughly $35°$. If two gears are to mesh without backlash, then they must use the same pressure angle. At one time, $14.5°$ was a commonly used pressure angle, but $20°$ is the current standard. A more obviously important choice is the number of teeth, N. Since the number of teeth determines the ratio of any gear train, this is a crucial choice, but it raises the question of how to fit N teeth on a given gear.

The tooth-to-tooth distance, as measured along the arc of the pitch circle, is the *circular pitch*, p_c, and the corresponding diameter is the *pitch diameter*, d. Since Np_c is the circumference of the pitch circle, we have

$$p_c = \frac{\pi d}{N}.$$

This is the inches or mm per tooth, measured along the circumerence. In practice, AGMA gears are specified by the *diametral pitch*,

$$p_d = \frac{N}{d}.$$

This is the teeth per π inches, and seems like an odd choice, but that's how it's done. Metric gears are specified by their *module*, m, which is stated in millimeters, and is

$$m = \frac{d}{N} = \frac{1}{p_d}.$$

The actual tooth-to-tooth distance is thus πm. The values around which the two systems, ISO and AGMA, are standardized are not compatible. For example, a module of $m = 4$ corresponds to a diametral pitch of

$$p_d = \frac{25.4}{4} = 6.35,$$

which is not a standard AGMA size.

The profile of each gear tooth is an involute, and determining the involute requires that the base circle be known. See Figure (10), in which the outer circle is the pitch circle and the inner circle is the base circle. Let r_p be the radius of the pitch circle, and r_b be the radius of the base circle. Because the angle determined by where the line of action meets the base circle is equal to the pressure angle, ϕ, we have

$$r_b = r_p \cos\phi.$$

The module or diametral pitch determines the tooth-to-tooth distance, but it doesn't determine how much of that space is solid tooth and how much is the space between teeth. The ensure that there is no backlash, the thickness of each tooth, as measured along the arc of the pitch circle, should be equal to the space between teeth – for practical reasons (lubrication), the gap between teeth is often made one or two thousandths of an inch wider than this. Again,

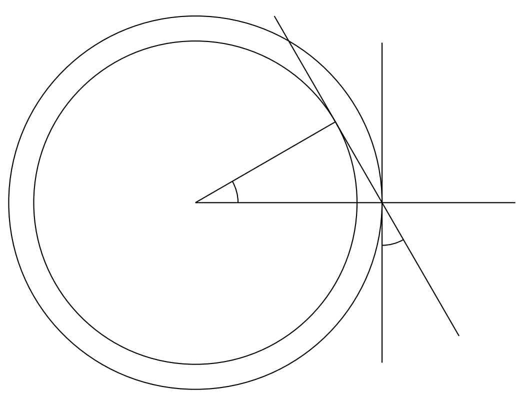

Figure 10: Base Circle from Pitch Circle

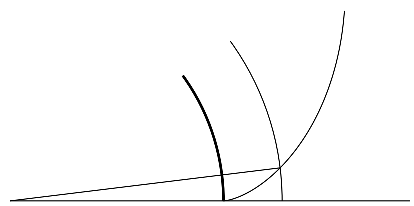

Figure 11: Angle Subtended by Involutes.

these distances are *as measured along the arc of the pitch circle.* In practice, it is easier to work with the angles subtended by these arcs.

There is one further issue to resolve. See Figure (11), which shows a portion of a base circle and slightly larger pitch circle, with an involute. What's needed is the measure of the small angle relative to the x-axis at which the involute meets the pitch circle. The involute is parametrized by

$$i(t) = r_b(\cos t + t\sin t, \sin t - t\cos t) = (i_x(t), i_y(t)),$$

and the involute meets the pitch circle when $|i(t)|^2 = r_p^2$. We have

$$\begin{aligned} |i(t)|^2 &= r_b^2\left[(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2\right] \\ &= r_b^2\left[\cos^2 t + 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t - 2t\cos t\sin t + t^2\cos^2 t\right] \\ &= r_b^2(1+t^2), \end{aligned}$$

and $|i(t)| = r_p$ implies that

$$t = \sqrt{\left(\frac{r_p}{r_b}\right)^2 - 1} = \sqrt{\left(\frac{r_p}{r_p\cos\phi}\right)^2 - 1} = \tan\phi.$$

The angle made by the line through $i(t)$ with the x-axis is α, where[1]

$$\tan\alpha = i_y(t)/i_x(t).$$

We now have enough information to begin laying out gear profiles. Suppose that ϕ, N and m (or p_d) are given. There will be N involutes runing one way, and N running the other way. Relative to the base circle, the tooth-to-tooth distance subtends an angle measuring $2\pi/N$. Half of this is solid tooth, and half is the gap between teeth, with an adjustment for α. So each solid tooth subtends the angle $\pi/N + 2\alpha$, and each gap subtends the angle $\pi/N - 2\alpha$. Figure (12) shows the result for $\phi = 20°$, $N = 15$ and $m = 4$.

[1] There seems to be no standard notation for this angle. In fact, I have found no mention of this issue in any common reference, even though it's crucial for determining the profile.

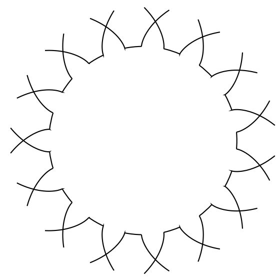

Figure 12: Basic Gear Form.

There is a glaring problem with Figure (12): the involutes continue beyond the point where the two sides of each tooth meet. If the aim is to program a milling machine to cut these profiles, then that's not a big deal – the machine will be cutting a bit of air beyond the end of each tooth – but it would be nice to know exactly where the two sides meet. Suppose that a tooth is symmetric about the x-axis so that the two sides meet at $y = 0$. Let R_θ be the rotation matrix through angle θ. Then the involute below the x-axis is parameterized by $R_{-\theta}i(t)$, where $\theta = \alpha + \pi/2N$. In particular, we want to find t such that the y-coordinate of $R_{-\theta}i(t)$ is equal to zero. We have

$$R_{-\theta}\ i(t) \quad = \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r_b(\cos t + t\sin t) \\ r_b(\sin t - t\cos t) \end{pmatrix},$$

and we require t such that

$$-r_b \sin\theta(\cos t + t\sin t) + r_b\cos\theta(\sin t - t\cos t) = 0$$

or

$$\frac{\sin t - t\cos t}{\cos t + t\sin t} = \tan\theta.$$

Unfortunately, finding such t requires the use of numerical methods of approximation. Making use of something like Newton-Raphson to determine t, we obtain Figure (13).

Figure (13) still doesn't look quite right. Gears don't typically have such pointy-ended teeth, and the gaps between the teeth don't seem deep enough in Figure (13). There are two more parameters to adjust for this: the *addendum*, a, and *dedendum*, b. The addendum is the distance above the pitch circle to which the teeth extend; when a tooth reaches a radius of $r_p + a$, it is truncated and given a flat top (the so-called *top land*). The dedendum is the depth below the pitch circle to which the gap between teeth is cut; so the gaps are cut to a radius of $r_p - b$ (forming the so-called *bottom land*). While the parameters a and b could take any value, they have been standardized to

$$a = m \qquad \text{and} \qquad b = 1.25\ m.$$

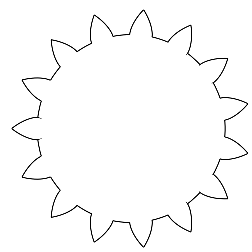

Figure 13: Corrected Gear Form.

Under the AGMA system (inches), these are

$$a = 1/p_d \qquad \text{and} \qquad b = 1.25/p_d.$$

Teeth have been standardized this way because the tips of pointy-ended teeth are prone to burring, while cutting the gaps more deeply allows for fuller engagement of the teeth. Figure (14) shows the same gear as in Figure (13), but with the addendum and dedendum circles.

It is now possible to specify the standard tooth profile for a gear with arbitrary parameters, as in Figure (15). The value for t to which the parameterization of the involute extends must be adjusted. Instead of extending out to the value of t_0 for which

$$\frac{\sin t_0 - t_0 \cos t_0}{\cos t_0 + t_0 \sin t_0} = \tan \theta,$$

t must be chosen so that $|i(t)| = r_p + a$. This is simpler to determine since θ no longer plays a role. As in an earlier calculation, we must have

$$\begin{aligned}(r_p + a)^2 &= |i(t)|^2 \\ &= r_b^2(1 + t^2)\end{aligned}$$

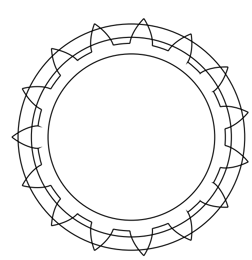

Figure 14: Gear with Addedenum and Dedendum Circles.

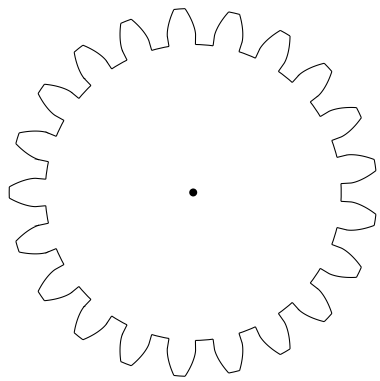

Figure 15: Standard Gear Profile.

or

$$t = \sqrt{\left(\frac{r_p + a}{r_b}\right)^2 - 1} = \sqrt{\left(\frac{r_p + a}{r_p \cos\phi}\right)^2 - 1}.$$

Of course, a can't be chosen to produce a value for t larger than t_0.

When drawing gears with a given addendum, it can be useful to know the angle subtended by the top land. As noted above, the angle subtended relative to the base circle by each tooth is $\pi/N + 2\alpha$. Let t_d be the value for t at which the top land begins. Each side of the tooth, from the base circle to the top land subtends the angle β, where $\tan\beta = i_y(t_d)/i_x(t_d)$. The top land thus subtends the angle $\pi/N + 2\alpha - 2\beta$.